Entropy and Futures Contracts

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Introduction Different futures products show markedly different distributions of activity across contract months. For some products, trading is concentrated almost exclusively in the front-month contract; in other products many different contracts are active on any given day. Futures contracts fall into three robust categories:

A. Traded volume is concentrated in the front month, as for the 10-year Treasury futures contract (Figure 3). Products in category A have all their traded volume concentrated in the single front-month maturity, except for brief well-structured roll periods in which volume is briefly split between the front month and the deferred month. Government debt, equity index, and foreign exchange products belong to this category.

B. Traded volume is divided among a handful of maturities, often with seasonal effects such as a concentration in the December contract. Rolls are complex events, as trade activity shifts among a collection of different maturities. Examples are crude oil (Figure 4) and corn (Figure 5). Energy futures and agricultural futures both belong to category B.

C. Traded volume is spread across many maturities . Products in category C are generally short-term interest rate (STIR) futures such as Eurodollars (Figure 6).

Our goal is to systematize and quantify this categorization. For example, are these three categories distinct, or are contracts placed along a continuous range from one active maturity to many? Do the categories as described above align with product class? We find that there is a surprisingly neat separation of products under these categories, and that with the possible exception of precious metals and the Canadian Bankers' Acceptance (BAX) futures, these categories align well with product class.

We use intraday and end-of-day data obtained from CME, Eurex, ICE Europe (formerly LIFFE), and Montréal exchanges, looking at all products in which our firm has consistently traded significant volumes. The data shown in Figure 1 and Figure 2 were taken from across one year, from July 2014 through June 2015. Table 1 shows summary statistics over the same period. The products are in six distinct classes: equity index, energy, foreign exchange (FX), metal, agricultural, and interest rate (which we subdivide into STIR and government debt).

Class	Product	Electronic Symbol	Bloomberg Symbol	ADV (Thousands)
STIR	Eurodollar	GE	ED	2,488
	Three-Month Sterling	L	L	490
	Three-Month Euro EURIBOR	I	ER	337
	Canadian Bankers' Acceptance	BAX	BA	94
Government Debt	US Ten-Year Note	ZN	TY	1,140
	US Five-Year Note	ZF	FV	713
	Euro-Bund	FGBL	RX	678
	Euro-Bobl	FGBM	OE	410
	US Two-Year Note	ZT	TU	293
	US Thirty-Year Bond	ZB	US	292
	Euro-Schatz	FGBS	DU	228
	Euro-BTP	FBTP	IK	84
	US Ultra Bond	UB	WN	76
	Euro-OAT	FOAT	OAT	72
	Canadian Ten-Year Note	CGB	CN	66
	Euro-Buxl	FGBX	UB	31
	Short-Term Euro-BTP	FBTS	BTS	18
Equity Index	E-mini S&P 500	ES	ES	1,467
	Euro STOXX 50	FESX	VG	1,038
	E-mini NASDAQ 100	NQ	NQ	253
	DAX Index	FDAX	GX	121
	FTSE 100 Index	Z	Z	112
	S&P Toronto 80	SXF	PT	19
	E-mini S&P Midcap 400	EMD	FA	17
Energy	WTI Crude Oil	CL	CL	631
	Natural Gas	NG	NG	267
	RBOB Gasoline	RB	XB	121
	Heating Oil	HO	HO	116
Foreign Exchange	Euro	6E	EC	241
	Japanese Yen	6J	JY	151
	British Pound	6B	BP	98
	Australian Dollar	6A	AD	96
	Canadian Dollar	6C	CD	63
	Mexican Peso	6M	PE	42
	Swiss Franc	6S	SF	32
	New Zealand Dollar	6N	NV	19
Metal	Gold	GC	GC	151
	Copper	HG	HG	56
	Silver	SI	SI	37
Agricultural	Corn	ZC	C	263
	Soybeans	ZS	S	184
	Wheat	ZW	W	105
	Soybean Oil	ZL	BO	91
	Soybean Meal	ZM	SM	77
	KC Wheat	KE	KW	24

Table 1: Futures products traded on CME, Eurex, ICE Europe (LIFFE), and Montréal exchanges. Average daily volume is measured across one year from July 2014 through June 2015. Colors are those used in Figures 1 and 2. We have divided interest rate futures into short term interest rate (STIR) and government debt.

Diversity measures Suppose that $p_1, \dots, p_n$ are a partition across n bins, with each $0 \le p_i \le 1$ and $p_1 + \cdots + p_n = 1$. In our application, the p_i are the fractions of traded volume in each of n futures maturities. We want to calculate a *diversity index* or *effective number* describing how "spread-out" these values are.

Diversity measures have been widely studied in ecology and in economics. A useful family can be constructed as follows, tracing the analysis of Hill [1973] and Jost [2006]. For $p = (p_1, \dots, p_n)$ with restrictions on p_i as above, and for $q \ge 0$, we write the *generalized mean* as

$$M_q(p) \;=\; \langle p^q \rangle^{1/q} \;=\; \left(\sum_i p_i \, p_i^q \right)^{1/q},$$

where $\langle x \rangle = \sum p(x) x$ defines the standard mean. In particular, $M_1(p) = \langle p \rangle$, $M_\infty(p) = \max p_i$, and $M_0(p) = \exp \sum_i p_i \log p_i$ (in the limit $q \to 0$). Then the *effective number of order q* is

$$N_q(p) \;=\; \frac{1}{M_{q-1}(p)}.$$

If p_i has the value $1/k$ for k different indices and is zero elsewhere, so that the weight is spread evenly across k different maturities, then $N_q(p) = k$ for $q > 0$. This motivates the use of the term "effective number."

It is easy to see that $M_q(p)$ is increasing in q and hence N_q is decreasing in q. Smaller values of q assign a more uniform weight to elements of the partition, giving relatively more weight to less common elements. In the limit $q \to 0$, N_q converges to the total number of nonzero p_i.

Two values of q in particular are in widespread use:

- $N_1(p) = \exp \sum_i -p_i \log p_i$ is the exponential of the Shannon entropy, originally developed for use in signal processing (Shannon [1948]).
- $N_2(p) = 1/\sum_i p_i^2$ is directly related to the Simpson index in ecology and to the Herfindahl-Hirschman index in economics, after Simpson [1949] and Hirschman [1945] respectively. The latter is used by the US Justice Department to evaluate the monopoly risk of proposed mergers.

For our study we prefer N_1, because of its theoretical elegance and because of its focus on the broad distribution.

We therefore fix our definition of effective number

$$N_{\text{eff}}(p) \;=\; \exp \sum_i -p_i \log p_i.$$

This definition systematizes our intuitive ideas about the number of actively traded contracts of a futures product, extending our ability to rank products by the diversity of their trades.

In this study we apply this construction to daily traded volume, normalizing the values on each day so that the values for all contracts on the same product sum to 1. For the purple lines in Figures 3–6, we plot N_{eff} as the value on each day. The range of observed values of N_{eff} for each product can be seen in Figure 1.

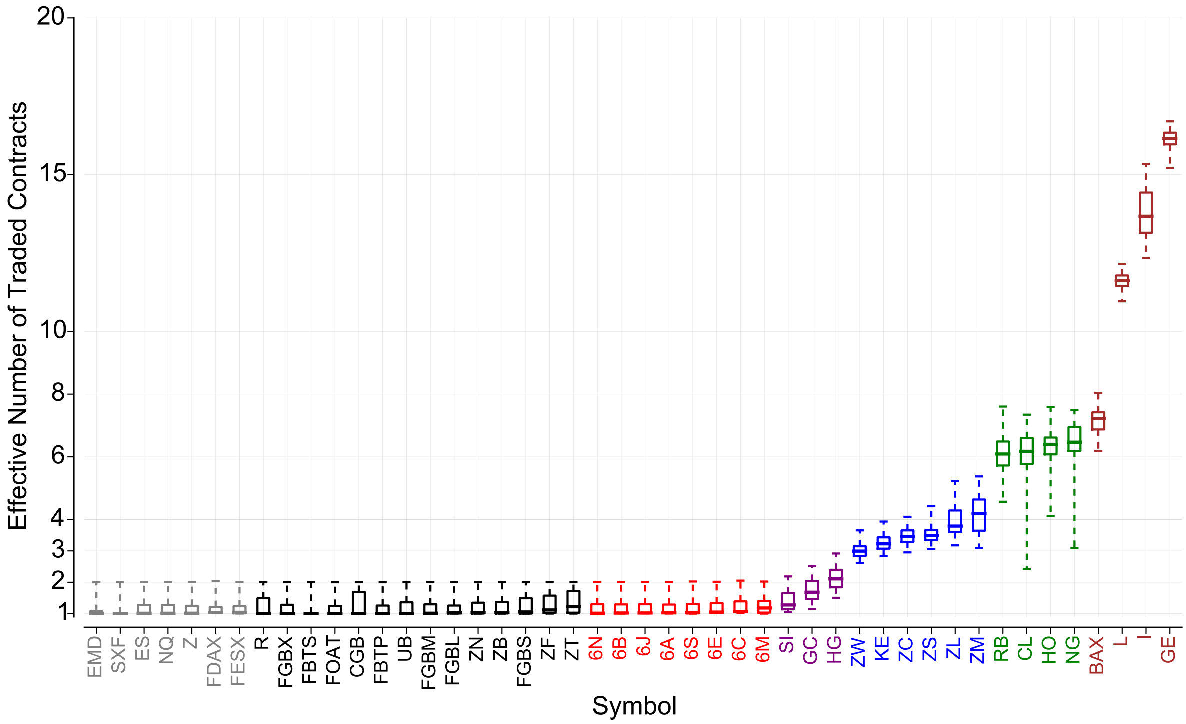

Figure 1: Effective number of daily traded contracts for each product considered. For each product we show the median (bold midline), center 50% interval (box), and total range (whiskers) for the daily value of N_{eff} for that product.

Empirical Findings As shown in Figure 2, government debt, foreign exchange, equity index, and precious metal products all trade a very small effective number of contracts. These category A products are all almost entirely traded on just one or two contracts, but show significant variation in their average daily traded volume, as shown by the spread along the vertical axis. Further right on the plot, category B energy and agricultural products tend to be traded on 3–7 contracts, indicating a more gradual roll period. Finally, the short-term interest rate products tend to trade a high effective number of contracts, indicating that these category C products spread their trading activity relatively evenly over the available contracts.

Detailed Examination We can improve our understanding further by examining detailed day-by-day graphs of daily traded volume and entropy. Data is compiled from daily and end-of-day data from CME, Eurex, ICE Europe (LIFFE), and Montréal exchanges. In all of these graphs, the daily traded volume is shown by green or gold lines and measured in thousands of lots, and the effective number of traded contracts is shown in purple.

Figure 3 shows the 10-year Treasury futures contract (ZN), as an example of a "typical" roll structure. This pattern is typical of government debt, equity index, and

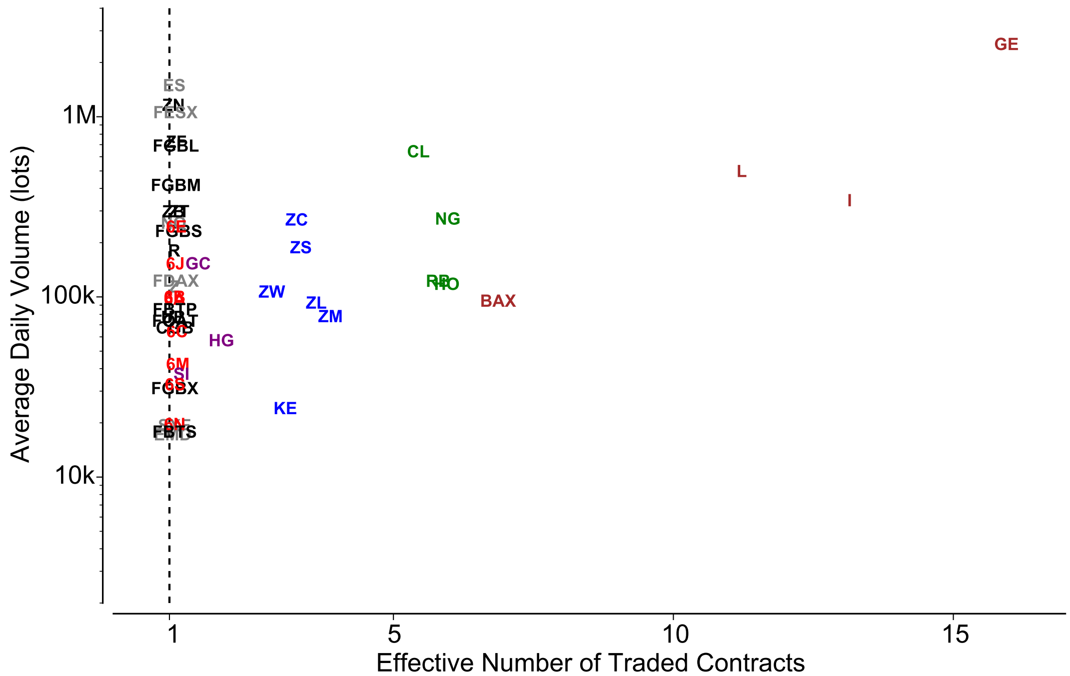

Figure 2: Effective number versus average daily volume. Category A products have an effective number close to one, and cluster near the left of the graph. Other classes of instrument can be accurately distinguished by their effective number. The vertical dashed line marks an effective number of 1, which is the lowest value possible.

foreign exchange (FX) futures. On most dates, all the trade volume is concentrated in the single front month contract. The entropy is near zero and the effective number of contracts is one. Only around the roll periods is traded volume split between two contracts. The effective number of contracts rises as high as two.

Figures 4 and 5 show the WTI Crude Oil (CL) and Corn (ZC) futures. These products show an intermediate roll structure, which is typical of energy and agricultural contracts. In this structure, trade volume is divided among a handful of different maturities, but divided in a predictable way which repeats itself from one roll period to the next. In these plots, December contract volumes are shown in gold to highlight the seasonal variation in trading patterns.

Figure 6 shows the Eurodollar complex, which is typical of STIR products in having its traded volume distributed across a wide range of product maturities.

Conclusion By evaluating the entropy of a product's traded contracts, we can quantitatively categorize the nature of that product's roll-structure. This categorization also strongly aligns with the class of the asset underlying the futures contracts. This quantification of the roll-structure yields a useful tool for understanding the diverse dynamics of different futures markets.

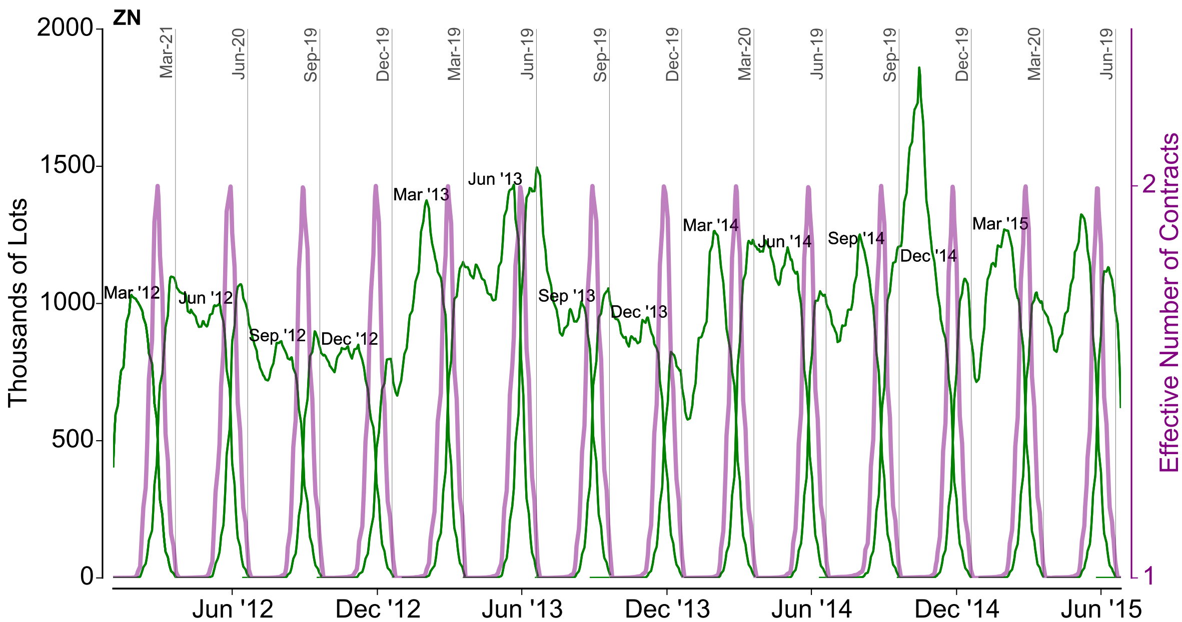

Figure 3: Daily traded volume and entropy of 10-year Treasury futures. Daily traded volumes are shown in green, and effective number of contracts shown in purple. Grey vertical lines mark contract expiration dates.

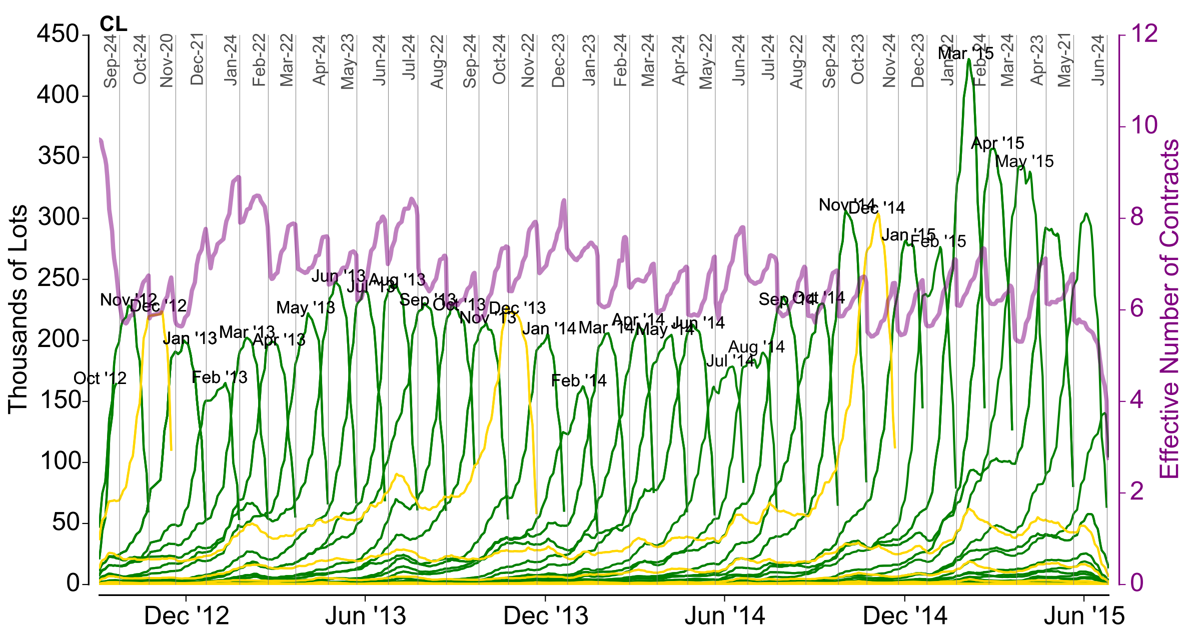

Figure 4: Daily traded volume and entropy of crude oil futures. Daily traded volumes are shown in green (December contracts in gold), and effective number of contracts shown in purple. Grey vertical lines mark contract expiration dates.

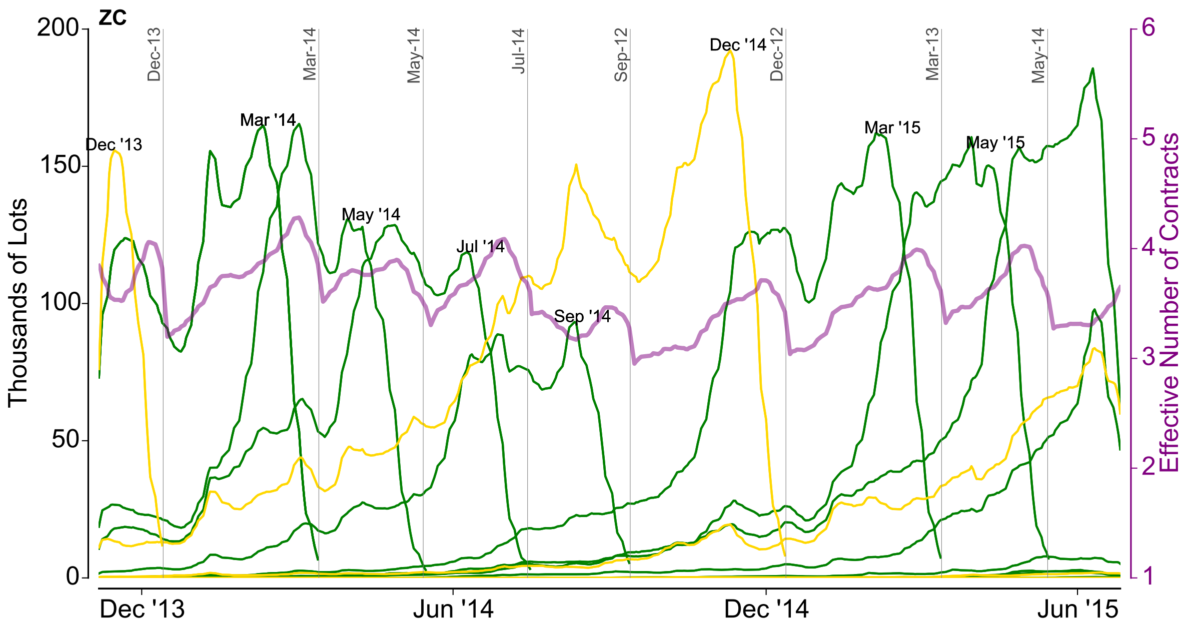

Figure 5: Daily traded volume and entropy of corn futures. Daily traded volumes are shown in green (December contracts in gold), and effective number of contracts shown in purple. Grey vertical lines mark contract expiration dates.

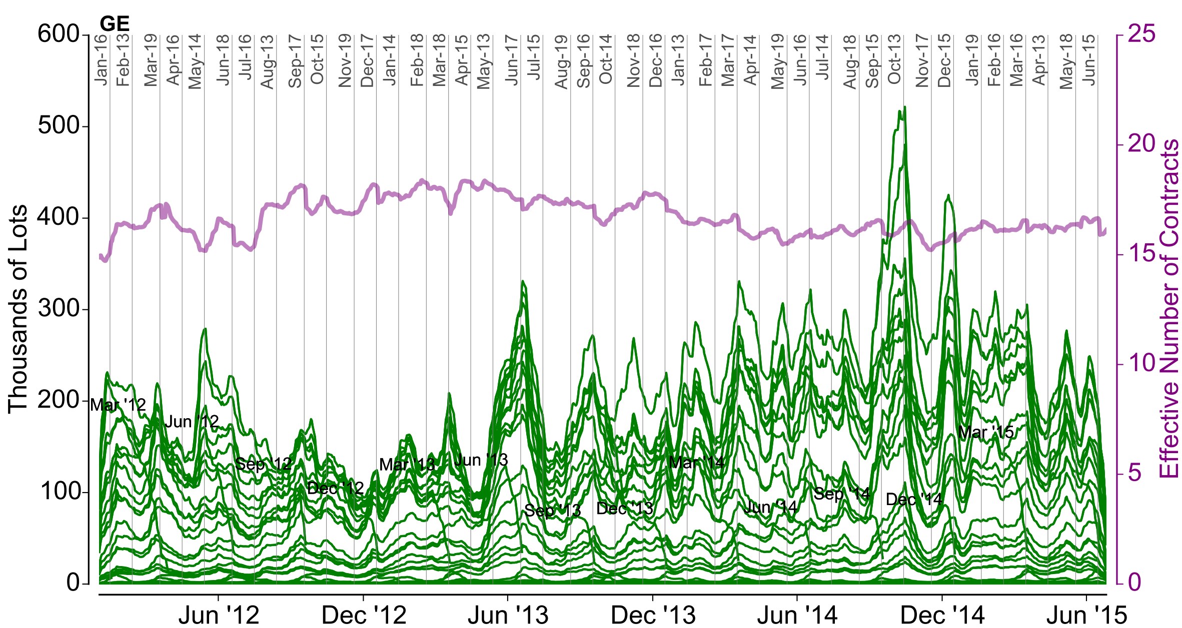

Figure 6: Daily traded volume and entropy of Eurodollar futures. Daily traded volumes are shown in green, and effective number of contracts shown in purple. Grey vertical lines mark contract expiration dates.

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